Stable decompositions of countable equivalence relations

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Notations for today:

- (X,μ) : standard probability space
- $\mathcal{R}$: countable ergodic pmp equivalence relation
- $[\mathcal{R}]$ = "full group of $\mathcal{R}$"

$$= \{T \in \mathrm{Aut}(X,\mu) \mid \forall^* x \in X : T(x)\,\mathcal{R}\,x\}$$

Def: $\mathcal{R}$ is stable $\iff \mathcal{R} \cong \mathcal{R} \times \mathcal{R}_{hyp}$

e.g. $\mathcal{R}(\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \curvearrowright \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z})$

$= E_0$.

Thm (Jones-Schmidt '85) TFAE:

1) $\mathcal{R}$ is stable

2) $\exists\, T_n \in [\mathcal{R}],\ A_n \subseteq X,\ \mu(A_n) = \frac{1}{2}$:

- T_n is central: $\forall S \in [\mathcal{R}] : \mu(\{x \mid T_n S x \neq S T_n x\}) \to 0$.
- A_n are almost invariant:

$$\forall S \in [\mathcal{R}] : \mu(SA_n \,\Delta\, A_n) \to 0.$$

$\cdot\ T_n A_n \simeq X \setminus A_n.$

Question: R_1, R_2 not stable.

$$R_1 \times R_{hyp} \cong R_2 \times R_{hyp} \implies R_1 \cong R_2^t \ ?$$

If yes: "unique stable decomposition".

Thm (Ioana-S. '18) Yes if R_1 is strongly ergodic.

Def: R is Schmidt if $\exists\, T_n \in [R]$ central s.t. $\liminf_n \mu(\{x \mid T_n x \neq x\}) > 0$.

Rk: $\Gamma \curvearrowright X$ Schmidt $\Rightarrow$ Γ inner amenable.
$\exists\ \cancel{\Leftarrow}$?
open

Thm (S. '21) Suppose R_1 not Schmidt, $R_1 \times R_{hyp} \cong R_2 \times R_{hyp}$. Then either

1) R_2 is not Schmidt, and $R_1 \cong R_2^t$.

2) R_2 is stable, i.e. $R_2 \cong R_2 \times R_{hyp} \cong R_1 \times R_{hyp}$.

1) Schmidt Property.

Lemma (S. '21) If R is not Schmidt, then $\exists F \Subset [[R]]$, $\exists K>0$ s.t. $\forall v \in [[R]]$

$$\mu(\{x \mid v(x) \neq x\}) \leq K \sum_{w \in F} \mu(\{x \mid vw(x) \neq wv(x)\})$$

Here: $[[R]] = \{ v : \underset{X}{\overset{}{A}} \xrightarrow{\sim} \underset{X}{B} \mid \forall^{*} x \in A : v(x) R x \}$. ($A \subseteq X$, $B \subseteq X$)

Lemma (S. '21) If R is not Schmidt and $v_n \in [[\overset{X}{R} \times \overset{Y}{S}]]$ is central, then $\exists X \ni x \longmapsto v_{n,x} \in [[S]]$ s.t.

$$(\mu \times \nu)(\{(x,y) \in X \times Y \mid v_n(x,y) \neq (x, v_{n,x}(y))\}) \to 0$$

Lemma (S. '21) Assume $\overset{X_1}{R_1} \times \overset{Y_1}{R_{hyp}} \cong \overset{X_2}{R_2} \times \overset{Y_2}{R_{hyp}}$, R_1 not Schmidt. Then either

- R_2 is not Schmidt, or
- R_2 is stable

"Proof": Assume R_2 is Schmidt, not stable:

- $\exists T_n \in [R_2]$ central s.t.
  - $\mu_2(\{x \in X_2 \mid T_n x \neq x\}) = 1$.
  - $\forall A_n \subseteq X_2$ alm. inv.: $\mu_2(T_n A_n \Delta A_n) \to 0$.

$\rightsquigarrow$ $T_n \times id \in [R_2 \times R_{hyp}] = [R_1 \times R_{hyp}]$.

$\overset{\text{Lemma}}{\Longrightarrow}$ $(x, T_{n,x})$

$\Rightarrow$ $T_{n,x}$ "is in the tail of R_{hyp}"

$\Rightarrow$ construct A_n s.t.

$$\mu((T_n \times id) A_n \Delta A_n) \not\to 0. \quad ↯$$

$\square$

2) Intertwining

Lemma (Ioana '11) $S, T \leq R$. TFAE

1) $\exists \underset{B}{\cancel{E}} \subseteq X$, $T_0 \leq T$, $\forall \underset{B_0}{\cancel{E_0}} \subseteq \underset{B}{\cancel{E}}$ $\exists Y \subseteq \underset{B_0}{\cancel{E_0}}$

   $\theta : Y \to Z \in [[R]]$:

   a) $T_0|_Y \leq T|_Y$ has finite index.

   b) $(\theta \times \theta)(T_0|_Y) \leq S|_Z$.

2) $\nexists\, \theta_n \in [T] : \varphi_S(\psi \theta_n \psi') \to 0 \quad \forall \psi, \psi' \in [R]$

   where $\varphi_S(\theta) = \mu(\{x \mid \theta(x) \, S \, x\})$

Notation: $T \prec_R S$.

Thm (S.'21) Assume $R_1 \times S_1 \cong R_2 \times S_2$

- $R_1 \prec R_2 \Rightarrow \exists T \leq R_2 : R_2 \cong R_1 \times T$.

(cf. [Ioana–S.'18])

- $R_1 \prec R_2$ and $R_2 \prec R_1$

  $\Rightarrow$ same is true for S_i's

  & $R_1 \cong R_2^t$, $S_1 \cong S_2^s$.

Lemma: $R_1 \times R_{hyp,1} \cong R_2 \times R_{hyp,2}$

R_1 not Schmidt.

$\Longrightarrow R_{hyp,2} \lesssim R_{hyp,1}$.